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Spin 2 Field Equation in Expanding Universe

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Abstract The spin 2 field equations are separated in the Robertson-Walker space-time by the Newman-Penrose formalism and by using a null tetrad frame previously considered. The angular and radial separated equations are integrated by generalizing and improving results relative to the massless case. The separated time equations are governed by two coupled linear differential equations that depend on the cosmological background. They are solved and studied for some models of cosmological expansion such as the linear and exponential expansion and the matter dominated and radiative expansion of the standard cosmology.

Keywords Spin 2 field equation · R-W Space-time · Variable separation · Solutions

1 Introduction

Now-a-days a unified and systematic formulation of massive field equations in curved space-time can be considered a well established result (e.g. [4] and references therein). As a consequence many physical considerations can be developed at the level of a general space-time [4, 11]. This is not the case for the explicit solution of the equations that seems in general a difficult task. Solution for special values of the spin and for particular space-time model do however exist. The separation of the Dirac equation in Kerr metric, originally obtained in [3], is a prototype for subsequent results. The separation method has been variously developed (e.g. [5, 6] and references therein; [17, 18]). Only to mention results concerning space-time of physical interest, we recall that the field equations of arbitrary spin can be separated by variable separation in the Robertson-Walker space-time [17] (as well as in the

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Schwarzschild space-time [18]). The proof of this result does not involve the explicit solution of the separated equations that however had been determined for the massive spin $1/2, 1, 3/2$ and for the massless spin 2 field equation [13, 15–17]. To obtain the general solution, one is tempted to proceed by induction on the value of the spin by taking into account the recurrence structure of the detailed equations as it results in [17]. However this appears a complex and cumbersome problem. In any case it seems not useless to proceed to solve the equations case by case, by fixing the value of the spin.

The object of the present paper is therefore of proceeding in the line of the previous results by explicitly solving the spin 2 field equation in the Robertson-Walker space-time. In this way one completes and extends the study of the massless equation. This last equation, that formally coincides with the Bianchi identity in vacuum [10], has been formulated and solved in terms of one only spinor field in [17]. The solution of the massive case gives also, in principle, the possibility of determining normal modes of the field in view of a quantization of the theory [2].

The spin 2 field equations are formulated here in the line proposed in [4]. They are expressed by two coupled spinor equations in a pair of spinor fields with suitable symmetry properties. The equations are first explicited in the Newman-Penrose formalism [8] by using a null tetrad frame previously considered. There result a system of sixteen coupled linear differential equations in the directional derivatives and spin coefficients. These equations are separated by applying the separation method of [17]. With a suitable choice of the separation constants, the angular and radial equations are reported to those of the massless case that are improved and discussed. The main difference with respect to the treatment of [14] lies in the separated time equations. Here the time evolution is not governed by one only equation, but by two coupled linear differential equations whose coefficients depend on the cosmological background. The time equations are integrated explicitly for the massless field case, in the static Universe case and, due to the physical interest, for a linear and an exponential expansion law and for the matter dominated and radiative expansion of the Standard Cosmology.

2 Separation of Spin 2 Equation in Robertson-Walker Space-Time

According to the formulation proposed in [4] the spin 2 field equation in spinor form can be written in a general conformally flat space-time as.

$$\nabla^{A}_{\dot{X}}\phi_{AA_1A_2A_3} + \mu_\star \chi_{A_1A_2A_3\dot{X}} = 0 \tag{1a}$$

$$\nabla^{\dot{Z}}_{A}\chi_{A_1A_2A_3\dot{Z}} - \mu_\star \phi_{AA_1A_2A_3} = 0 \tag{1b}$$

$\nabla_{A\dot{X}}$ is the spinor covariant derivative, $\mu_\star = \frac{im_0 c}{\sqrt{2}\hbar}$ (we set $c = \hbar = 1$ in the following), m_0 being the mass of the particles of the field. The spinor fields are assumed to have the symmetry properties $\phi_{AA_1A_2A_3} = \phi_{(AA_1A_2A_3)}$ and $\chi_{A_1A_2A_3\dot{X}} = \chi_{(A_1A_2A_A3)\dot{X}}$. The object is now to prove that the solutions of the (1) can be determined in the context of the Robertson-Walker space-time whose line element is given by

$$ds^2 = dt^2 - R(t)^2\left[\frac{dr^2}{1-ar^2} + r^2(d\theta^2 + \sin\theta^2 d\varphi^2)\right], \quad a = 0, \pm 1. \tag{2}$$

The space-time model is of interest, because the Robertson-Walker metric is the base for the Standard Cosmology. On account of the symmetry properties of the spinors it is useful to set:

$$\begin{aligned} \phi_h \equiv \phi_{AA_1A_2A_3} \quad &\Leftrightarrow \quad A + A_1 + A_2 + A_3 = h, \quad h = 0, 1, 2, 3, 4 \\ \chi_{k\dot{X}} \equiv \chi_{A_1A_2A_3\dot{X}} \quad &\Leftrightarrow \quad A + A_1 + A_2 = k, \quad k = 0, 1, 2, 3 \end{aligned} \tag{3}$$

To separate (1) we first develop the scheme in the Newman-Penrose formalism [8] by assuming the null tetrad frame e_a^μ $(a = 1, 2, 3, 4, \mu = t, r, \theta, \varphi)$, whose corresponding directional derivatives and non zero spin coefficients are given by [13]

$$\begin{aligned} D &= \partial_{0\dot{0}} = e_1^\mu \partial_\mu = \frac{1}{\sqrt{2}}\left(\partial_t + \frac{\sqrt{1-ar^2}}{R}\partial_r\right) \\ \Delta &= \partial_{1\dot{1}} = e_2^\mu \partial_\mu = \frac{1}{\sqrt{2}}\left(\partial_t - \frac{\sqrt{1-ar^2}}{R}\partial_r\right) \\ \delta &= \partial_{0\dot{1}} = e_3^\mu \partial_\mu = \frac{1}{rR\sqrt{2}}(\partial_\theta + i\csc\theta\,\partial_\varphi) \\ \delta^\star &= \partial_{1\dot{0}} = e_4^\mu \partial_\mu = \frac{1}{rR\sqrt{2}}(\partial_\theta - i\csc\theta\,\partial_\varphi) \end{aligned} \tag{4}$$

$$\begin{aligned} \rho &= -\frac{1}{\sqrt{2}}\left(\frac{\dot{R}}{R} + \frac{\sqrt{1-ar^2}}{rR}\right), \qquad \epsilon = -\gamma = \frac{\dot{R}}{2\sqrt{2}R} \\ \mu &= \frac{1}{\sqrt{2}}\left(\frac{\dot{R}}{R} - \frac{\sqrt{1-ar^2}}{rR}\right), \qquad \alpha = -\beta = \frac{\cot\theta}{2rR\sqrt{2}}, \end{aligned} \tag{5}$$

$(\dot{R} = dR/dt)$. By expliciting the spinorial derivatives one then obtains eight plus eight equations corresponding to (1a), (1b) respectively, in terms of directional derivatives and spinor coefficients:

$$\begin{aligned} (D - 4\rho - 2\epsilon)\phi_1 - (\delta^\star + 4\beta)\phi_0 &= \mu_\star \chi_{0\dot{0}} \\ (D - 3\rho)\phi_2 - (\delta^\star + 2\beta)\phi_1 &= \mu_\star \chi_{1\dot{0}} \\ (D - 2\rho + 2\epsilon)\phi_3 - \delta^\star\phi_2 &= \mu_\star \chi_{2\dot{0}} \\ (D - \rho + 4\epsilon)\phi_4 - (\delta^\star - 2\beta)\phi_3 &= \mu_\star \chi_{3\dot{0}} \\ (\Delta + \mu + 4\epsilon)\phi_0 - (\delta - 2\beta)\phi_1 &= -\mu_\star \chi_{0\dot{1}} \\ (\Delta + 2\mu + 2\epsilon)\phi_1 - \delta\phi_2 &= -\mu_\star \chi_{1\dot{1}} \\ (\Delta + 3\mu)\phi_2 - (\delta + 2\beta)\phi_3 &= -\mu_\star \chi_{2\dot{1}} \\ (\Delta + 4\mu - 2\epsilon)\phi_3 - (\delta + 4\beta)\phi_4 &= -\mu_\star \chi_{3\dot{1}} \end{aligned} \tag{6a}$$

$$
\begin{aligned}
(D-\rho-2\epsilon)\chi_{0\dot{1}}-(\delta-2\beta)\chi_{0\dot{0}} &= -\mu_\star\phi_0 \\
(D-\rho)\chi_{1\dot{1}}-\delta\chi_{1\dot{0}}+\mu\chi_{0\dot{0}} &= -\mu_\star\phi_1 \\
(D-\rho+2\epsilon)\chi_{2\dot{1}}-(\delta+\beta)\chi_{2\dot{0}}+2\mu\chi_{1\dot{0}} &= -\mu_\star\phi_2 \\
(D-\rho+4\epsilon)\chi_{3\dot{1}}-(\delta+4\beta)\chi_{3\dot{0}}+3\mu\chi_{2\dot{0}} &= -\mu_\star\phi_3 \\
(\Delta+\mu+4\epsilon)\chi_{0\dot{0}}-(\delta^\star+4\beta)\chi_{0\dot{1}}-3\rho\chi_{1\dot{1}} &= \mu_\star\phi_1 \\
(\Delta+\mu+2\epsilon)\chi_{1\dot{0}}-(\delta^\star+2\beta)\chi_{1\dot{1}}-2\rho\chi_{2\dot{1}} &= \mu_\star\phi_2 \\
(\Delta+\mu)\chi_{2\dot{0}}-\delta^\star\chi_{2\dot{1}}-\rho\chi_{3\dot{1}} &= \mu_\star\phi_3 \\
(\Delta+\mu-2\epsilon)\chi_{3\dot{0}}-(\delta^\star-2\beta)\chi_{3\dot{1}} &= \mu_\star\phi_4
\end{aligned}
\tag{6b}
$$

Noting that the spin coefficients do not depend on the variable φ, (6) can be separated elementally by setting

$$
\begin{aligned}
\phi_h(t,r,\theta,\varphi) &= \alpha(t)\phi_h(r)S_h(\theta)\exp(im\varphi), \quad h=0,1,2,3,4 \\
\chi_{j\dot{0}}(t,r,\theta,\varphi) &= A(t)\phi_{j+1}(r)S_{j+1}(\theta)\exp(im\varphi), \\
\chi_{j\dot{1}}(t,r,\theta,\varphi) &= -A(t)\phi_j(r)S_j(\theta)\exp(im\varphi), \quad j=0,1,2,3
\end{aligned}
\tag{7}
$$

In view of a possible determination of the normal modes or by continuity assumptions in $\theta=0,\pi$ it is convenient to assume $m=0,\pm1,\pm2,\pm3,\ldots$. By using the expressions (7) and (5), equations (6) can be separated. The eight angular equations one finally obtains are given by

$$
L^-_{2-j}S_j=\lambda_j S_{j+1}, \qquad L^+_{j-1}S_{j+1}=\lambda_{4+j}S_j, \quad j=0,1,2,3 \tag{8}
$$

where $L^\pm_d=\partial_\theta\mp\csc\theta+d\cot\theta$, λ_i, $i=0,1,2,\ldots,7$ being the angular integration constants. These equations come from the separation of (6a). The separation of (6b) gives rise again to (8) after suitably identifying the integration constants with the λ_i.

Instead the separated radial equations are given by the eight coupled equations

$$
\begin{aligned}
ik\phi_{j+1} &= \sqrt{1-ar^2}\phi'_{j+1}+\frac{4-j}{r}\sqrt{1-ar^2}\phi_{j+1}-\frac{\lambda_j}{r}\phi_j, \\
ik\phi_j &= -\sqrt{1-ar^2}\phi'_j-\frac{j+1}{r}\sqrt{1-ar^2}\phi_j-\frac{\lambda_{4+j}}{r}\phi_{j+1}, \quad j=0,1,2,3
\end{aligned}
\tag{9}
$$

that are consistently obtained by identifying the separation constants relative to the separation of the r,t variables with the one only constant k.

Finally the separated time equations come out to be

$$
\dot{\alpha}R+3\alpha\dot{R}-im_0RA=-ik\alpha, \qquad \dot{A}R-im_0R\alpha=ikA \tag{10}
$$

There results that the main difference with respect to the massless spin 2 field equation as treated in [14] is that one has now to deal with a pair of coupled equations instead that with one only equation.

3 Angular and Radial Equations

If one chooses the angular integration constants to be related by

$$
\lambda_0\lambda_4=\lambda_1\lambda_5+2, \qquad \lambda_1\lambda_5=\lambda_2\lambda_6, \qquad \lambda_2\lambda_6=\lambda_3\lambda_7-2 \tag{11}
$$

then the system of (8) can be reduced to five independent equations each one in one only of the angular functions S_i's. These equations have been already solved in [14] under the condition of regularity in $\theta = 0, \pi$. The result is that, by setting $\lambda^2 = -\lambda_0\lambda_4$, one finds

$$\lambda^2 + 2 = l(l+1), \quad l = 2, 3, 4, \dots \tag{12}$$

and the $S_i = S_{ilm}(\theta)$, $i = 0, 1, 2, 3, 4$, $l \geq |m|$. The solutions S_{0lm}, S_{1lm} are essentially given in terms of Jacobi polynomials [1] and S_{2lm} in terms of Legendre functions. It further results $S_{3lm} \cong S_{1l-m}$, $S_{4lm} \cong S_{0l-m}$. The explicit expressions of the S_{ilm}'s are given in [14].

One can note that non trivial S_i's exist when all the λ_i's vanish. Indeed in that case the system (8) admits the (not necessarily regular) solution

$$\begin{aligned} S_0 &= \sin^2\theta/(\tan\theta)^m \\ S_1 &= S_3 = 0 \\ S_4 &= \sin^2\theta(\tan\theta)^m \end{aligned} \tag{13}$$

with $S_2 = 0$ if $m \neq 0$ and $S_2 = const \neq 0$ if $m = 0$.

Also the system of the eight radial equation (9) can be reduced [14] to five independent disintangled equations each one in one only of the radial function $\phi_i(r)$. From those equations there results that one can choose $\phi_4 \cong \phi_0^\star$, $\phi_3 \cong \phi_1^\star$ so that one is left with the equations for ϕ_0, ϕ_1, ϕ_2. These equation are difficult to be solved exactly in the open and closed ($a = \pm 1$) space-time case. Instead in the flat case $a = 0$, they can be compactly written

$$\phi_d'' + \frac{6}{r}\phi' + \left[k^2 + \frac{4-\lambda^2}{r^2} + 2(2-d)\frac{ik}{r}\right]\phi_d = 0, \quad d = 0, 1, 2\ (a = 0) \tag{14}$$

By setting $\phi_d(r) = r^{l-2}\exp(ikr)Z_d(r)$ into (14) one finds that $Z_d(\xi)$, $\xi = -2ikr$, satisfies the confluent hypergeometric equation

$$\xi Z_d'' + (2l + 2 - \xi)Z_d' - (l + 3 - d)Z_d = 0, \quad d = 0, 1, 2 \tag{15}$$

Therefore a solution is $\phi_d = r^{l-2}\exp(ikr)\Phi(l-d+3; 2l+2; -2ikr)$, $d = 0, 1, 2$. (A second linearly independent solution has a complicated structure due to the fact that $2l + 2$ is a positive integer [12].) By taking into account the asymptotic behaviour of the confluent hypergeometric function [1] one can note that

$$\begin{aligned} \phi_0 &\overset{r\to\infty}{\longrightarrow} \left(\frac{i}{2k}\right)^{l-1}\frac{(2l+1)!}{(l+2)!}\frac{e^{-ikr}}{r} \\ \phi_1 &\overset{r\to\infty}{\longrightarrow} \left(\frac{i}{2k}\right)^{l}\frac{(2l+1)!}{(l+1)!}\frac{e^{-ikr}}{r^2} \\ \phi_2 &\overset{r\to\infty}{\longrightarrow} 2\left(\frac{1}{2k}\right)^{l+1}\frac{(2l+1)!}{l!}\frac{\cos[kr - \frac{\pi}{2}(l+1)]}{r^3} \end{aligned} \tag{16}$$

while for small distances

$$\phi_d \overset{r\to 0}{\longrightarrow} r^{l-2}, \quad d = 0, 1, 2 \tag{17}$$

Therefore the radial functions are all bounded in $r = 0$ for $l \geq 2$.

Note that also the system of equations (9) admits for $\lambda_i = 0\ \forall i$, the particular solution

$$\phi_1 = \phi_2 = \phi_3 = 0, \qquad \phi_0 = \phi_4^\star = \frac{e^{-ikr}}{r} \tag{18}$$

that, together with (13), furnishes a special solution $\phi_h(t, r, \theta, \varphi)$, $\chi_{k\dot{X}}(t, r, \theta, \varphi)$ of (1).

4 Separated Time Equations

The coupled equations (10) can be easily solved in the massless case and in the static universe case.

Suppose $m_0 = 0$. Then from (10) one obtains

$$\begin{aligned} A(t) &= A(0)\exp\left(ik\int_0^t \frac{dt'}{R(t')}\right) \\ \alpha(t) &= \alpha(0)\frac{R^3(0)}{R^3(t)}\exp\left(-ik\int_0^t \frac{dt'}{R(t')}\right) \end{aligned} \tag{19}$$

The expression for $\alpha(t)$ coincides with the corresponding expression of [14], where the massless field was described in terms of one only spinor fields.

Let now $R(t) = const = R_0$. The solution is

$$\begin{aligned} A(t) &= A(0)\exp(\pm i\omega t), \quad \omega = \sqrt{m_0^2 + (k/R_0)^2} \\ \alpha(t) &= \frac{A(0)}{m_0 R_0}(\pm\omega R_0 - k)\exp(\pm i\omega t) \end{aligned} \tag{20}$$

The coupled equations (10) are difficult to be solved in general for an arbitrary cosmological background $R(t)$. It is possible to study them for special expansions law of the universe that are of physical interest.

Linear expansion $R = Ht$, $H = const$. This corresponds to the curvature dominated expansion for the Friedmann equation with $p = -\rho/3$ [7, 9]. From the system (10) one can obtain a closed equation for $A(t)$ that results, in the present case, to be

$$\ddot{A} + \frac{3}{t}\dot{A} + \left(m_0^2 + \frac{k^2 - 2ikH}{H^2t^2}\right)A = 0 \tag{21}$$

By setting $A = t^\alpha \exp(im_0 t)Z(t)$ into (21) and then $\xi = -2im_0 t$ one is left with the confluent hypergeometric equation for $Z(\xi)$:

$$\xi Z'' + (2\alpha + 3 - \xi)Z' - \frac{5}{2}Z = 0, \quad \alpha = i\frac{k}{H}, -2 - i\frac{k}{H} \tag{22}$$

A solution for A is therefore given by $A(t) = t^\alpha \exp(im_0 t)\Phi(\frac{5}{2}; 2\alpha + 3; -2im_0 t)$. (A second independent solution can be obtained in standard way since $2\alpha + 3$ is not in general an integer number [12].) The solution for α then follows from the given $A(t)$ and the relation

$$\alpha(t) = -\frac{i}{m_0}\dot{A} - \frac{k}{Hm_0}\frac{A(t)}{t} \tag{23}$$

Exponential Expansion $R = \exp(Ht)$, $H = const$. This describes an inflationary phase or a vacuum dominated expansion of the Standard Cosmology [7]. By setting $\tau = \exp(-Ht)$ into (10), the differential equation for $A(\tau)$ results to be ($A' = dA/d\tau$):

$$A'' - \frac{2}{\tau}A' + \frac{m_0^2 - 2ikH\tau + k^2\tau^2}{\tau^2 H^2}A = 0 \tag{24}$$

If one defines $A = \tau^\alpha \exp(-ik\tau/H)Z(\tau)$ and $\xi = 2ik\tau/H$, then (24) gives for $Z(\xi)$ the confluent hypergeometric equation

$$\xi Z'' + (2\alpha - 2 - \xi)Z' - \alpha Z = 0, \quad \alpha = \frac{3}{2}\left(1 \pm \left[1 - \left(\frac{2m_0}{3H}\right)^2\right]^{\frac{1}{2}}\right) \tag{25}$$

Therefore a solution for A is

$$A(t) = \tau^\alpha \exp\left(-i\frac{k}{H}\tau\right)\Phi\left(\alpha; 2\alpha - 2; 2i\frac{k}{H}\tau\right), \quad \tau = \exp(-Ht) \tag{26}$$

The α solution follows then from (10), (26) through the expression

$$\alpha(t) = \frac{i}{m_0}\tau\left[HA'(\tau) + ikA(\tau)\right], \quad \tau = \exp(-Ht). \tag{27}$$

Matter Dominated Expansion $R \sim t^{2/3}$ The expansion law corresponds now to the zero pressure situation of the Standard Cosmology. To solve (10) it is useful to consider the conformal time $\tau : d\tau = dt/dR$ (e.g. [7]). In the present case one has $\tau \sim t^{1/3}$, $R = E\tau^2$, $E = const$. The system (10) reads now ($' = d/d\tau$)

$$\begin{aligned} \alpha' + \frac{6}{\tau}\alpha - im_0 E\tau^2 A &= -ik\alpha \\ A' - im_0 E\tau^2\alpha &= ikA \end{aligned} \tag{28}$$

The system (28) gives elementally

$$\begin{aligned} \alpha &= -\frac{\tau^2}{m_0 E}(kA - iA') \\ A'' + \frac{4}{\tau}A' + \left(m_0^2E^2\tau^4 + k - \frac{4ik}{\tau}\right)A &= 0 \end{aligned} \tag{29}$$

The equation for A is of the Fuchsian type whose solution is not standard. By general results, independent solutions are of the form $A_1 = \tau^{-3}\sum_0^\infty c_n\tau^n$, $A_2 = A_1 \log\tau + \sum_0^\infty d_n\tau^n$. The coefficients c_n can be determined by an elementary series integration of (29) while the d_n's have a rather complicated structure [12].

Radiation Dominated Standard Cosmology $R \sim t^{1/2}$ By proceeding as in the previous case, one has $t \sim \tau^2$, $R = d\tau$, d constant. The system (10) becomes now

$$\begin{aligned} \alpha &= -\frac{1}{m_0 d\tau}(kA + iA') \\ A'' + \frac{2}{\tau}A' + \frac{m_0^2 d\tau^3 + k^2\tau - 2ik}{\tau}A &= 0 \end{aligned} \tag{30}$$

Solutions for A can be obtained as in the previous case. By considering the indicial equation of the differential equation for A, they are $A_1 = \tau^{-1}\sum_0^\infty c_n\tau^n$, $A_2 = A_1 \log\tau + \sum_0^\infty d_n\tau^n$ with again the c_n's to be determined by series integration [12].

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